ME 314 - Engineering Design : Mechanical Components
Lecture 10

4-17 Stresses in (pressurized) Cylinders

Cylinders are often used as pressure vessels or pipelines and can be subjected to internal & external pressure. Examples are:

* Air or hydraulic cylinders
* Fluid storage tanks & pipes
* Gun barrels

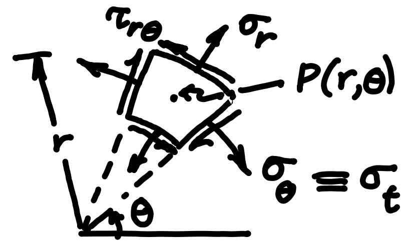
Stress Components in Polar Coordinates

Radial Stress = σ_r

Tangential Stress = $\sigma_\theta = \sigma_t$

Longitudinal (or Axial) stress = $\sigma_L = \sigma_a$

(σ_a is non-zero only when caps are present)



Stress state at the point $P(r,\theta)$

Thick-walled Cylinders

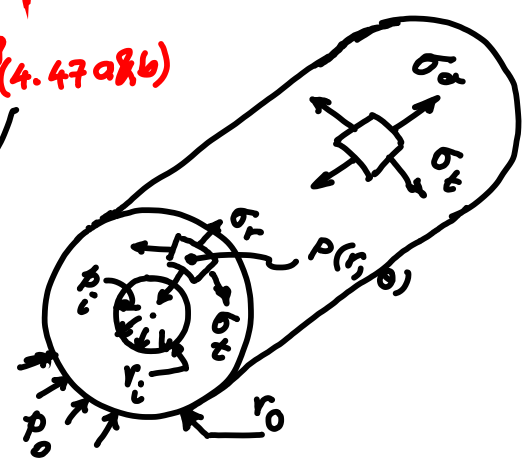
Hoop Stress

From theory of elasticity, we have

$$\sigma_t = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} + \frac{r_i^2 r_o^2 (p_i - p_o)}{r^2 (r_o^2 - r_i^2)}$$

$$\sigma_r = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} - \frac{r_i^2 r_o^2 (p_i - p_o)}{r^2 (r_o^2 - r_i^2)}$$

(4.47a&b)



(Thickness, $t = r_o - r_i > 0.1\, r_i$)

where p_0 and p_i are external and internal pressures; and r_0 and r_i are outer and inner radii, respectively.

If the ends of the cylinder are closed, i.e., it has a cap, the axial stress in the walls is

$$\sigma_a = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} \qquad (4.47c)$$

Note that σ_t and σ_r are functions of r, while σ_a is uniform across the thickness. Also note that $\tau_{r\theta} = 0$ so that stresses are all principal.

Cylinder under internal pressure only ($p_0 = 0$):

$$\sigma_t = \frac{r_i^2 p_i}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r^2}\right) \qquad (4.48a)$$

$$\sigma_r = \frac{r_i^2 p_i}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right) \qquad (4.48b)$$

Cylinder under external pressure only ($p_i = 0$):

$$\sigma_t = \frac{-r_o^2 p_o}{r_o^2 - r_i^2}\left(1 + \frac{r_i^2}{r^2}\right)$$

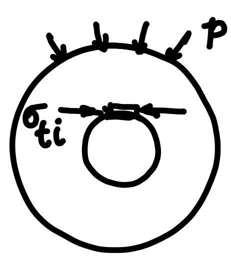
$$\sigma_r = -\frac{r_o^2 p_o}{r_o^2 - r_i^2}\left(1 - \frac{r_i^2}{r^2}\right)$$

Stresses at the inner & outer surfaces

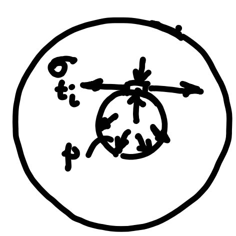
Stress	External Pressure only	Internal Pressure only
σ_{to}	$-\dfrac{(r_o^2 + r_i^2)p}{(r_o + r_i)t}$	$\dfrac{2r_i^2 p}{(r_o + r_i)t}$
σ_{ro}	$-p$	0
σ_{ti}	$-\dfrac{2r_o^2 p}{(r_o + r_i)t}$	$\dfrac{(r_o^2 + r_i^2)p}{(r_o + r_i)t}$
σ_{ri}	0	$-p$
τ_{max}	$\frac{1}{2}\sigma_{ti}$	$\frac{1}{2}(\sigma_{ti} + p)$

Remark: Note that for both cases (external and internal pressure), $|\sigma_{ti}|$ is the maximum stress.

Ext. Pressure only



Int. Pressure only



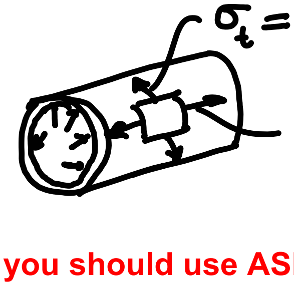
Example 1. An aluminum cylinder of internal diameter 0.742 in and an outer diameter of 1.486 in is under an external pressure of 400 psi. Determine the maximum stress developed.

Example 2. A steel pipe with a 1.75"- internal diameter is under an internal pressure of 2,000 psi. Assuming that the allowable stress for the steel pipe is 20,000 psi, determine the outer diameter.

Thin-walled Cylinders (t < 0.10 r)

In this case, the stress distribution can be assumed to be uniform across the thin wall and the above expressions simplify to

$$\sigma_t = \frac{pr}{t} = \text{Hoop Stress}$$



$$\sigma_a = \frac{pr}{2t}$$

Caution: For a safe design of pressure vessels, you should use ASME Boiler codes.

CHAPTER 8 - THE FINITE ELEMENT ANALYSIS

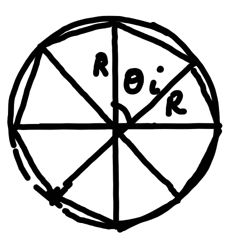
8.0 Introduction

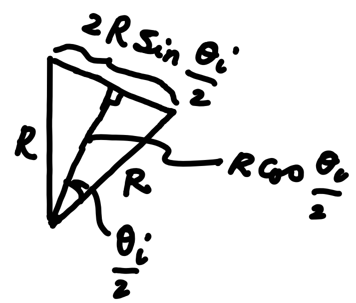
The *finite element method* (FEM) is a numerical procedure for analyzing parts and structures. It is used to solve problems that are too complicated and can not be solved by analytical methods. In general, FEM models a structure as an assemblage of small parts or *elements*. Elements are called *finite* to distinguish them from differential elements used in calculus. Each element has a simple geometry and therefore is much easier to analyze than the actual structure.

General Examples:

- **Lego (kids' play)**
- **Buildings**
- **Approximation of the area of a circle**

Let N = total number of triangles (elements)





Area of one triangle: $S_i = \frac{1}{2}\left(2R\sin\frac{\theta_i}{2}\right)\left(R\cos\frac{\theta_i}{2}\right)$

$$= \frac{1}{2}R^2\sin\theta_i$$

Area of all triangles: $S_N = \sum_{i=1}^{N} S_i = \frac{1}{2}R^2\sum_{i=1}^{N}\sin\theta_i$

$$= \frac{1}{2}R^2\left(\sin\theta_1 + \sin\theta_2 + \cdots + \sin\theta_N\right)$$

$$= \frac{1}{2}R^2 N\sin\frac{2\pi}{N}$$

Note:

$\theta_1 = \theta_2 = \cdots = \theta_N = \frac{2\pi}{N}$

Applications of FEM in Engineering

FEM/FEA is the most widely used computer simulation method in Engineering because it is integrated with CAD/CAM applications and it is much easier than hand calculations and experiments.

-FEM is used in stress analysis of solids

-Heat conduction problems

-Fluid mechanics

-Structural analysis

Historically …..

-Courant (1943)- variational methods

-Turner, Clough, Martin and Topp (1956) - stiffness

-Clough (1960) - "Finite Element" for 2D stress analysis problems

-Applications on mainframe computers (1970s)

-Applications on microcomputers, pre- and postprocessors (1980s)

-Analysis of large structural systems (1990s)

FEM in Structural Analysis

-Divide structure into pieces (elements with nodes)

-Describe the behavior of the physical quantities for each element

-Assemble these equations for the nodes to form an approximate system of equations for the whole structure

-Solve the system of equations involving unknown quantities at the nodes (e.g., displacements or forces)

-Calculate desired quantities (e.g., stresses and strains) at selected elements

-Check for errors by performing (convergence) test

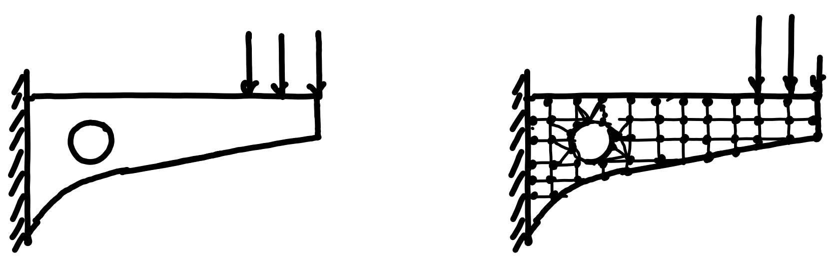
Consider the plane structure shown. Black dots, called ***nodes*** or ***node points***, indicate where elements are connected to one another. In this plane model, each node has two ***degrees of freedom*** (**d.o.f**), i. e., each node can displace in both x and y directions.

-If there are n nodes, there are $2n$ d.o.f. in the model.

-The actual structure has infinitely many particles so there are infinitely many d.o.f.

-The solution by FEM involves the solution of $2n$ algebraic equations for the $2n$ d.o.f.

-Use of only $2n$ d.o.f. in analysis is similar to use of the first $2n$ terms of a convergent infinite series.



8.1 One-Dimensional Spring System

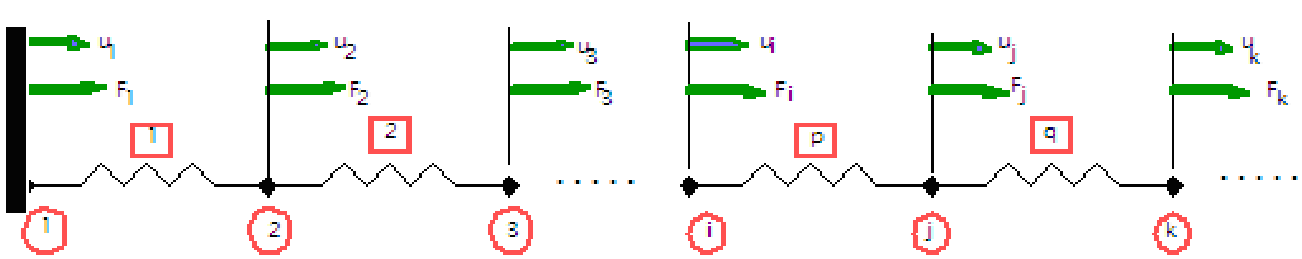
The finite element method is best illustrated by analysis of a one-dimensional spring system.

-Each spring is an element identified by the number in the box.

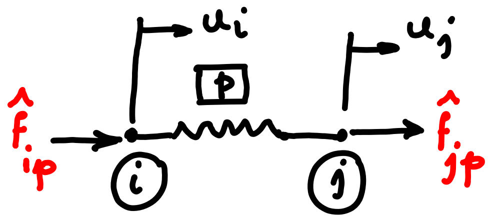
-The spring elements are connected at nodes.

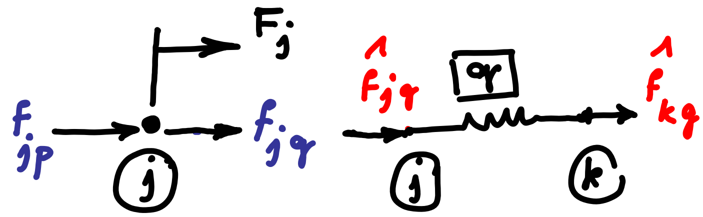
-Each node is identified by the number in the circle.

-The displacement of node i is denoted by u_i. The external force acting on node i is designated by F_i.



1. The Element Equation





$\hat{f}_{ip}$ **is the internal force acting on the element** ***p*** **due to node** ***i*** **displacement;**

f_{ip} **is the internal force acting on node** ***i*** **by element** ***p*****.**

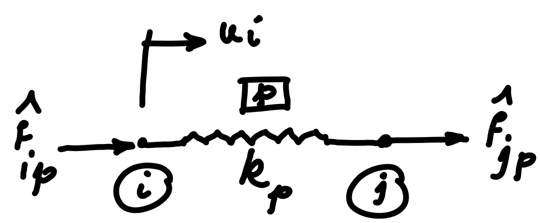
Newton's third law requires that

$$\hat{f}_{ip} = -f_{ip} \quad , \quad \hat{f}_{jp} = -f_{jp}$$

*** Suppose $u_i \neq 0$ but $u_j = 0$, then assuming that the spring is linear and requiring equilibrium**

$$\hat{f}_{ip} = k_p u_i$$

$$\hat{f}_{jp} = -\hat{f}_{ip} = -k_p u_i$$



*** Now suppose $u_i = 0$ but $u_j \neq 0$, then**

$$\hat{f}_{jp} = k_p u_j \quad , \quad \hat{f}_{ip} = -\hat{f}_{jp} = -k_p u_j$$

*** If both $u_i \neq 0$ and $u_j \neq 0$, then element forces are**

$$\left.\begin{aligned} \hat{f}_{ip} &= k_p u_i - k_p u_j \\ \hat{f}_{jp} &= -k_p u_i + k_p u_j \end{aligned}\right\}$$

* In terms of *nodal forces*:

$$\left.\begin{aligned} -f_{ip} &= k_p u_i - k_p u_j \\ -f_{jp} &= -k_p u_i + k_p u_j \end{aligned}\right\} \qquad (1)_1$$

In matrix form

$$\begin{bmatrix} k_p & -k_p \\ -k_p & k_p \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} -f_{ip} \\ -f_{jp} \end{Bmatrix} \qquad (1)_2$$

or

$$[k]\,\{d\} = \{f\} \qquad (1)_3$$

In this equation,

[*k*] is the *element stiffness matrix*
{*d*} is the *node displacement vector*
{*f*} is the *node internal force vector*

we refer to (1) as the *element equation*.

For element □, set $p = 1$, $i = 1$, and $j = 2$ in (1):

$$\left.\begin{aligned} k_1 u_1 - k_1 u_2 &= -f_{11} \\ -k_1 u_1 + k_1 u_2 &= -f_{21} \end{aligned}\right\} \quad \text{or} \quad \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -f_{11} \\ -f_{21} \end{Bmatrix} \qquad (2)$$

Similarly, for element □, set $p = 2$, $i = 2$, and $j = =3$ in (1):

$$\left.\begin{aligned} k_2 u_2 - k_2 u_3 &= -f_{22} \\ -k_2 u_2 + k_2 u_3 &= -f_{32} \end{aligned}\right\} \quad \text{or} \quad \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} -f_{22} \\ -f_{32} \end{Bmatrix} \qquad (3)$$